# Bernoulli differential equation

Given the following Bernoulli equation:

$$\frac{dy}{dx} + \frac{1}{x}y = -\frac{1}{x}y^2$$

a) Transform the equation into a linear one.
b) Solve the linear differential equation.
c) Find the general solution $y(x)$.

# Solution

Given the following Bernoulli equation:

$$\frac{dy}{dx} + \frac{1}{x}y = -\frac{1}{x}y^2$$

## a) Transform the equation into a linear one

The given equation is a Bernoulli equation of the form:

$$\frac{dy}{dx} + P(x)y = Q(x)y^n$$

where $P(x) = \frac{1}{x}$, $Q(x) = -\frac{1}{x}$, and $n = 2$.

To transform the equation into a linear one, we make the variable change:

$$v = y^{1-n} = y^{-1}$$

We differentiate $v$ with respect to $x$:

$$\frac{dv}{dx} = -y^{-2}\frac{dy}{dx} = -y^{-2}y'$$

We divide the original equation by $y^2$:

$$y'y^{-2} + \frac{y^{-1}}{x} = -\frac{1}{x}$$

Substituting $v$ and $v'$ in the original equation:

$$-v' + \frac{v}{x} = -\frac{1}{x}$$

Rearranging:

$$v' - \frac{v}{x} = \frac{1}{x}$$

This is a linear equation in terms of $v$.

## b) Solve the differential equation

Let $v = u \cdot w$, where $v' = u'w + w'u$. Substituting:

$$u'w + w'u - \frac{uw}{x} = \frac{1}{x}$$

Rearranging:

$$w(u' - \frac{u}{x}) - w'u = \frac{1}{x}$$

We solve two separate equations:

$$-w'u = \frac{1}{x}$$

and

$$u' - \frac{u}{x} = 0$$

$$du = \frac{u}{x}dx$$

$$\frac{du}{u} = \frac{dx}{x}$$

$$\ln(u) = \ln(x)$$

$$u = x$$

We substitute into the other equation and solve:

$$-w'x = \frac{1}{x}$$

$$-dw = \frac{1}{x^2}dx$$

$$-w = -x^{-1} + C$$

$$w = x^{-1} + C$$

The result is:

$$v = (x^{-1} + C)x = 1 + xC$$

$$y^{-1} = 1 + xC$$

## c) Find the general solution $y(x)$

Solving for $y$:

$$y = \frac{1}{1 + Cx}$$

where $C$ is an arbitrary constant of integration.